J Glob Optim (2008) 40:475–487
DOI 10.1007/s10898-007-9185-6

On the second conjugate of several convex functions in general normed vector spaces

Constantin Zălinescu

Received: 25 May 2007 / Accepted: 28 May 2007 / Published online: 27 June 2007

Abstract When dealing with convex functions defined on a normed vector space X the biconjugate is usually considered with respect to the dual system (X, X^*), that is, as a function defined on the initial space X. However, it is of interest to consider also the biconjugate as a function defined on the bidual X^{**}. It is the aim of this note to calculate the biconjugate of the functions obtained by several operations which preserve convexity. In particular we recover the result of Fitzpatrick and Simons on the biconjugate of the maximum of two convex functions with a much simpler proof.

Keywords Biconjugate · Composition with linear operators · Conjugate · Convex function · Convolution · Max-convolution · Maximum of convex functions · Normed vector space · Sum of convex functions

Mathematics Subject Classification (2000) 46N10 · 49J52 · 49N15

1 Introduction

When treating convex analysis in locally convex spaces it is natural to consider the dual system $(X, X^*, \langle\cdot,\cdot\rangle)$, where $\langle x, x^*\rangle := x^*(x)$ for $x \in X$ and $x^* \in X^*$. In the case when X is a normed vector space besides this dual system we can also consider the dual system $(X^*, X^{**}, \langle\cdot,\cdot\rangle)$, where $\langle x^*, x^{**}\rangle := x^{**}(x^*)$ for $x^* \in X^*$ and $x^{**} \in X^{**}$. Of course, in such a situation we identify X with $J(X)$, where $J : X \to X^{**}$, $\langle x^*, Jx\rangle := \langle x, x^*\rangle$; generally Jx is denoted by $\widehat{x}$ or simply x. In this way, having a function $g : X^* \to \overline{\mathbb{R}}$, for its conjugate one can consider both the dual system $(X, X^*, \langle\cdot,\cdot\rangle)$ as well as $(X^*, X^{**}, \langle\cdot,\cdot\rangle)$. In our book [9] we considered only the first situation. However it is interesting to consider also the second

C. Zălinescu (✉)
Faculty of Mathematics, University “Al.I.Cuza” Iaşi, 700506 Iaşi, Romania
e-mail: zalinesc@uaic.ro

C. Zălinescu
Romania and Institute of Mathematics Octav Mayer, Iaşi, Romania

situation. This was already done for example in [3,7,8]. More recently some problems in this context were studied in [2].

Our aim in this note is to give formulas for the second conjugate as a function defined on the bidual for several classes of convex functions like sums, compositions with linear operators, maximum, convolution, max-convolution and the new function introduced in [6]. In fact we shall see that these formulas can be easily deduced from the formulas for their conjugates already known.

2 Notation and preliminary results

Consider first a separated dual system $(X, Y, \langle\cdot,\cdot\rangle)$. This means that X, Y are real linear spaces and $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{R}$ is bilinear with the properties: $\langle x, y\rangle = 0$ for every $y \in Y$ implies $x = 0$ and $\langle x, y\rangle = 0$ for every $x \in X$ implies $y = 0$. Note that X and Y have symmetric roles. Having such a dual system on X we consider the weakest topology $\sigma(X, Y)$ which makes continuous all the functions $\langle\cdot, y\rangle$ with $y \in Y$, and similarly the topology $\sigma(Y, X)$ on Y. Also note that $(X, \sigma(X, Y))$ becomes a locally convex space whose topological dual can be, and is, identified with Y in the sense that for every continuous linear function $\varphi : (X, \sigma(X, Y)) \to \mathbb{R}$ there exists a unique $y \in Y$ such that $\varphi(x) = \langle x, y\rangle$ for every $x \in X$. A similar statement holds for $(Y, \sigma(Y, X))$. Starting with a separated locally convex space (X, τ) and taking X^* its topological dual, the natural dual system is $(X, X^*, \langle\cdot,\cdot\rangle)$ with $\langle x, x^*\rangle := x^*(x)$ for $x \in X$ and $x^* \in X^*$; we shall denote this dual system simply (X, X^*). In such a situation the topology $\sigma(X, X^*)$ is denoted by w and is called the weak topology on X, while $\sigma(X^*, X)$ is denoted by w^* and is called the weak star topology on X^*. Of course, $(X, w)^* = X^*$ and $(X^*, w^*)^* = X$ (by the above identifications). Remark that a general dual system $(X, Y, \langle\cdot,\cdot\rangle)$ is of the type $(X, X^*, \langle\cdot,\cdot\rangle)$ when we consider on X the topology $\tau = \sigma(X, Y)$ because in this case $X^* = Y$, as seen above. So, considering the framework of a separated locally convex space (X, τ) is sufficiently versatile. Even the algebraic case can be considered in this framework. For this consider a linear space E, its algebraic dual E' and $\langle x, \varphi\rangle := \varphi(x)$ for $x \in E$, $\varphi \in E'$. Then it is convenient to take on E also the core-topology τ_c, that is the locally convex topology generated by the family $\mathcal{P}$ of all the semi-norms defined on E. Any semi-norm $p \in \mathcal{P}$ is continuous with respect to (w.r.t. for short) τ_c and the interior w.r.t. τ_c of a convex set $A \subset E$ is nothing else than the algebraic interior (or core) of A; moreover, any convex function $f : E \to \overline{\mathbb{R}}$ is continuous on the core of its domain and f is subdifferentiable on the intrinsic core of its domain when proper. In this way all the results on convex functions stated in general locally convex spaces can be applied for convex functions defined on linear spaces without topology.

Let us recall some notions and notation for convex sets and functions defined on locally convex spaces (lcs for short). So let (X, τ) be a lcs and $A \subset X$. By $\operatorname{conv} A$, $\operatorname{aff} A$, A^i, ${}^iA$, $\operatorname{cl} A$ (or $\operatorname{cl}_\tau A$ or $\overline{A}$ or $\overline{A}^\tau$) we denote the convex hull, affine hull, algebraic interior (or core), relative algebraic interior (or intrinsic core), closure (with respect to τ when we want to emphasize the topology) of A; of course, $\overline{\operatorname{conv}} A = \operatorname{cl}(\operatorname{conv} A)$. Having $A \subset X$ the set ${}^{ic}A$ is ${}^iA$ when $\operatorname{aff} A$ is closed and ${}^{ic}A$ is the empty set in the contrary case. As usual, for $A, B \subset X$, $a \in X$, $\Gamma \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$ we set

$$A + B := \{a + b \mid a \in A,\ b \in B\}, \quad a + B := \{a\} + B,$$
$$\Gamma A := \{\gamma a \mid \gamma \in \Gamma,\ a \in A\}, \quad \alpha A := \{\alpha\}A, \quad -A := (-1)A.$$

When $A \subset X$ is a nonempty closed convex set, A_∞ is the asymptotic (or recession) cone of A, that is, the set $\bigcap_{t>0} t(A-a)$, where $a \in A$ (A_∞ does not depend on $a \in A$). Moreover, the indicator function associated to the set $A \subset X$ is the function $\iota_A : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ defined by $\iota_A(x) := 0$ for $x \in A$ and $\iota_A(x) := \infty$ for $x \in X \setminus A$, where $\infty := +\infty$.

Let $f : X \to \overline{\mathbb{R}}$; the domain and the epigraph of f are defined by

$$\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}, \quad \operatorname{epi} f := \{(x,t) \in X \times \mathbb{R} \mid f(x) \le t\}.$$

The function f is proper if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for every $x \in X$. The class of proper convex functions on X will be denoted by $\Lambda(X)$. To the function f we associate the greatest lower semicontinuous (lsc for short) function $\overline{f}$ or $\overline{f}^{\tau}$ majorized by f; hence $\operatorname{epi} \overline{f} = \operatorname{cl}(\operatorname{epi} f)$. When f is convex (which means that $\operatorname{epi} f$ is convex) then $\overline{f}$ is also convex. For an arbitrary function $f : X \to \overline{\mathbb{R}}$, $\overline{\operatorname{conv}} f$ is the function whose epigraph is $\overline{\operatorname{conv}}(\operatorname{epi} f)$ and is the greatest lsc convex function majorized by f. The class of proper lsc convex functions on X will be denoted by $\Gamma(X)$ or $\Gamma_\tau(X)$. The conjugate of $f : X \to \overline{\mathbb{R}}$ is the function

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x) \mid x \in X\} = \sup\{\langle x, x^*\rangle - f(x) \mid x \in \operatorname{dom} f\};$$

it is clear that $f^* = (\overline{f})^* = (\overline{\operatorname{conv}} f)^*$. Moreover, f^* is a w^*–lsc convex function; f^* is proper if and only if f is proper and minorized by a continuous affine function. It is clear that the supremum $\sup_{i \in I} f_i$ of a nonempty family of convex functions is convex, but its conjugate is not easy to compute. On the other hand the infimum $\inf_{i \in I} f_i$ of a family of convex functions is not convex, generally; however, $\inf_{i \in I} f_i$ is convex if $f_i \le f_j$ or $f_j \le f_i$ whenever $i, j \in I$. Using the definition of the conjugate one obtains easily the next formula for any functions $f_i : X \to \overline{\mathbb{R}}$ with $i \in I \neq \emptyset$:

$$(\inf_{i \in I} f_i)^* = \sup_{i \in I} f_i^*. \tag{1}$$

The ε-subdifferential of the proper function $f : X \to \overline{\mathbb{R}}$ at $x \in \operatorname{dom} f$ is defined by

$$\partial_\varepsilon f(x) := \{x^* \in X^* \mid \langle x' - x, x^*\rangle \le f(x') - f(x) + \varepsilon \ \forall x' \in X\},$$

where $\varepsilon \in \mathbb{R}_+ := [0, \infty)$; moreover, $\partial_\varepsilon f(x) := \emptyset$ for $x \in X \setminus \operatorname{dom} f$ and $\partial f(x) := \partial_0 f(x)$ for $x \in X$.

Let $f : X \to \overline{\mathbb{R}}$ and $A \in L(X, Y)$, that is, Y is another lcs and $A : X \to Y$ is a continuous linear operator. Define

$$Af : Y \to \overline{\mathbb{R}}, \qquad (Af)(y) := \inf\{f(x) \mid Ax = y\}$$

with the usual convention $\inf \emptyset := \infty$. Then $(Af)^* = f^* \circ A^*$, where $A^* : Y^* \to X^*$ is the adjoint of A (hence $\langle Ax, y^*\rangle = \langle x, A^* y^*\rangle$ for all $x \in X$ and $y^* \in Y^*$). Moreover, if f is convex then Af is convex, too. However, Af might be non lower semicontinuous even if f is lsc. Having the functions $f_1, \ldots, f_n : X \to \overline{\mathbb{R}}$, and using the conventions

$$0 \cdot \infty := \infty, \quad 0 \cdot (-\infty) := 0, \quad (-\infty) + (+\infty) := (+\infty) + (-\infty) := \infty,$$

the convolution and the max-convolution of $f_1, \ldots, f_n$ are the functions $f_1 \Box \ldots \Box f_n$, $f_1 \Diamond \cdots \Diamond f_n : X \to \overline{\mathbb{R}}$ defined by

$$(f_1 \Box \cdots \Box f_n)(x) := \inf\{f_1(x_1) + \ldots + f_n(x_n) \mid x_i \in X,\ x_1 + \ldots + x_n = x\},$$
$$(f_1 \Diamond \cdots \Diamond f_n)(x) := \inf\{f_1(x_1) \vee \cdots \vee f_n(x_n) \mid x_i \in X,\ x_1 + \ldots + x_n = x\},$$

where $a \vee b := \max\{a, b\}$ for $a, b \in \overline{\mathbb{R}}$; we say that the (max-) convolution is exact if the infimum in its definition is attained when finite. Observe that

$$a \vee b = \sup\{\lambda a + (1-\lambda)b \mid \lambda \in (0,1)\} \quad \forall a, b \in (-\infty, \infty]. \tag{2}$$

We have that $f_1 \Box \cdots \Box f_n = Ah$ and $f_1 \Diamond \cdots \Diamond f_n = Ak$, where $h, k : X^n \to \overline{\mathbb{R}}$ are defined by

$$h(x_1, \ldots, x_n) := f_1(x_1) + \ldots + f_n(x_n), \quad k(x_1, \ldots, x_n) := f_1(x_1) \vee \cdots \vee f_n(x_n), \tag{3}$$

and

$$A : X^n \to X, \quad A(x_1, \ldots, x_n) := x_1 + \ldots + x_n. \tag{4}$$

Hence $f_1 \Box \cdots \Box f_n$ and $f_1 \Diamond \cdots \Diamond f_n$ are convex when $f_1, \ldots, f_n$ are convex, but they are not necessarily lsc when $f_1, \ldots, f_n$ are lsc. Since $A^* x^* = (x^*, \ldots, x^*)$ one obtains $(f_1 \Box \cdots \Box f_n)^* = f_1^* + \ldots + f_n^*$.

We denote by $g^\star$ the conjugate of $g : X^* \to \overline{\mathbb{R}}$ w.r.t. the dual system (X, X^*), that is,

$$g^\star : X \to \overline{\mathbb{R}}, \qquad g^\star(x) := \sup\{\langle x, x^* \rangle - g(x^*) \mid x^* \in X^*\}.$$

An important result in convex analysis, the biconjugate theorem, asserts that for the proper function $f \colon X \to \overline{\mathbb{R}}$, $f = f^{*\star}$ if and only if $f \in \Gamma(X)$. It follows that for a function $f : X \to \overline{\mathbb{R}}$ with nonempty domain we have $f^{*\star} = \overline{\mathrm{conv}}\, f$ if $\overline{\mathrm{conv}}\, f$ is proper and $f^{*\star} = -\infty$ (that is the constant function $-\infty$) if $\overline{\mathrm{conv}}\, f$ is not proper; in particular, if f is convex, then $f^{*\star} = \overline{f}$ if $\overline{f}$ is proper and $f^{*\star} = -\infty$ otherwise. Applying the previous discussion for the dual system $(X^*, X, \langle\cdot,\cdot\rangle)$ if $g : X^* \to \overline{\mathbb{R}}$ is a proper convex function we have that $g^{\star *} = \overline{g}^{w^*}$ if $\overline{g}^{w^*}$ is proper and $g^{\star *} = -\infty$ otherwise.

These simple observations yield the following result.

Proposition 1 *Let X, Y be separated locally convex spaces and $A \in L(X, Y)$.*

(i) *If $h \in \Gamma(Y)$ is such that* $\mathrm{dom}\, h \cap \mathrm{Im}\, A \neq \emptyset$ *then* $(h \circ A)^* = \overline{A^* h^*}^{w^*}$.
(ii) *If $f_1, \ldots, f_n \in \Gamma(X)$ are such that* $\cap_{i \in \overline{1,n}} \mathrm{dom}\, f_i \neq \emptyset$ *then* $(f_1 + \ldots + f_n)^* = \overline{f_1^* \Box \cdots \Box f_n}^{w^*}$.

Proof (i) Of course, $h \circ A \in \Gamma(X)$, and so $(h \circ A)^{*\star} = h \circ A$. The function $A^* h^*$ is convex; moreover, for the dual system $(X^*, X, \langle\cdot,\cdot\rangle)$ we have that $(A^* h^*)^\star = h^{*\star} \circ (A^*)^* = h \circ A \in \Gamma(X)$, and so $(h \circ A)^* = (A^* h^*)^{\star *} = \overline{A^* h^*}^{w^*}$.

(ii) Taking h and A as in (3) and (4), respectively, we have that $h \in \Gamma(X^n)$ and $\mathrm{dom}\, h \cap \mathrm{Im}\, A \neq \emptyset$. The conclusion follows from (i). $\Box$

Note that assertion (i) of the preceding proposition is equivalent to [4, Th. 2.7] while assertion (ii) is equivalent to that in [4, Rem. 2.8]. Of course, instead of the topology w^* on X^* one can take any compatible topology with the dual system $(X^*, X, \langle\cdot,\cdot\rangle)$, that is, a locally convex topology σ on X^* for which the topological dual of X^* is X.

The lower semicontinuity of the functions in the preceding result is essential. Indeed, take $X = \mathbb{R}$, $f_1 := \iota_{(-\infty,0]}$, $f_2(x) := 0$ for $x > 0$, $f_2(0) := 1$, $f_2(x) := \infty$ for $x < 0$. Then $f_1 + f_2 = \iota_{\{0\}} + 1$, $(f_1 + f_2)^* = -1$, $f_1^* = \iota_{[0,\infty)} = \overline{f_2}$, $f_2^* = \iota_{(-\infty,0]} = f_1$, $f_1^* \Box f_2^* = 0$.

Also the conditions $\mathrm{dom}\, h \cap \mathrm{Im}\, A \neq \emptyset$ in (i) and $\cap_{i \in \overline{1,n}} \mathrm{dom}\, f_i \neq \emptyset$ in (ii) are essential. For this take $X = \mathbb{R}^2$, $f_1(x, y) := x^{-1}$ for $x > 0$, $f_1(x, y) := +\infty$ for $x \leq 0$ and $f_2 := \iota_{(-\infty,0] \times \mathbb{R}}$. It is clear that $\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 = \emptyset$, $f_1^*(x^*, y^*) = -2\sqrt{-x^*}$ for $x^* \leq 0$ and $y^* = 0$, $f_1^*(x^*, y^*) = \infty$ otherwise and $f_2^* = \iota_{[0,\infty) \times \{0\}}$. Moreover,

$(f_1^*\Box f_2^*)(x^*, y^*) = -\infty$ for $x^* \in \mathbb{R}$ and $y^* = 0$, $(f_1^*\Box f_2^*)(x^*, y^*) = \infty$ otherwise, and so $-\infty = (f_1 + f_2)^* \neq \overline{f_1^*\Box f_2^*}^{w^*} = \overline{f_1^*\Box f_2^*} = f_1^*\Box f_2^*$.

As in [7], we consider the right multiplication of the function $f : X \to \overline{\mathbb{R}}$ with the scalar $\lambda \in \mathbb{P} := (0, \infty)$ as being the function $f\lambda : X \to \overline{\mathbb{R}}$ defined by $(f\lambda)(x) := \lambda f(\lambda^{-1}x)$. Also, as in [7], when $f \in \Gamma(X)$, we set $f0 := f_\infty$, where the recession function f_∞ has as epigraph the recession cone of epi f; alternatively,

$$f_\infty(x) = \lim_{t\to\infty} t^{-1} f(x_0 + tx) = \sup\left\{t^{-1}[f(x_0 + tx) - f(x_0)] \mid t \in \mathbb{P}\right\},$$

where $x_0 \in \operatorname{dom} f$. An easy calculation shows that

$$(\alpha f)^* = f^*\alpha, \quad (f\alpha)^* = \alpha f^* \quad \forall \alpha \in \mathbb{P}. \tag{5}$$

Moreover (see f.i. [9, Exer. 2.23]),

$$\iota_{\operatorname{dom} f}^* = (0f)^* = f^*0, \quad (f0)^* = \overline{0f^*}^{w^*} = \iota_{\overline{\operatorname{dom} f^*}^{w^*}} \quad \forall f \in \Gamma(X). \tag{6}$$

In the next sections we consider X to be a normed vector space. Of course, X^* is also a normed vector space (even a Banach space) endowed with the dual norm. So, for a function $g : X^* \to \overline{\mathbb{R}}$ we can consider the conjugate $g^\star : X \to \overline{\mathbb{R}}$ corresponding to the dual system (X, X^*) or the conjugate $g^* : X^{**} \to \overline{\mathbb{R}}$ corresponding to the dual system (X^*, X^{**}). Taking into account the biconjugate theorem we have that $f^{**}|_X = f$ for every $f \in \Gamma(X)$. Our interest is to calculate the biconjugate h^{**} for several classes of convex functions $h : X \to \overline{\mathbb{R}}$ defined on a normed vector space X.

3 The biconjugates of convolutions

In the sequel X, Y are normed vector spaces if not mentioned otherwise; as usual X^*, Y^* are their topological duals and X^{**}, Y^{**} their topological biduals. The topologies $\sigma(X^*, X)$ and $\sigma(X^{**}, X^*)$ will be denoted by w^*. First of all, applying [9, Th. 2.4.14] to the function $s_A := \iota_A^*$, where $A \subset X$ is a nonempty convex set, for the dual system (X^*, X^{**}) we obtain (see [2, Sect. 4]) that

$$(\iota_A)^{**} = \iota_{\overline{J(A)}^{w^*}}. \tag{7}$$

A useful property of the biconjugate is obtained using the following result stated in [2, Lem. 4.5].

Proposition 2 *Let $y \in X$, $f \in \Gamma(X)$, $x^* \in \operatorname{dom} f^*$, $\gamma \geq 0$ and $x^{**} \in \partial_\gamma f^*(x^*)$ be such that $\|\widehat{y} - x^{**}\| < \alpha$. Then for every $\delta > 0$ and every w^*-neighborhood V of x^{**} in X^{**}, there exists $x_{\delta,V} \in X$ such that $\left\|y - x_{\delta,V}\right\| < \alpha$ and $\widehat{x_{\delta,V}} \in V \cap \partial_{\gamma+\delta} f^*(x^*)$.*

Using this result we got in [2, Cor. 4.7] the following extension of [5, Lem. 3.1]. Note that [5, Lem. 3.1] is proved for X a Banach space and using the formula $(f \vee g)^{**} = f^{**} \vee g^{**}$ obtained in [7] when f is continuous at some point of $\operatorname{dom} f \cap \operatorname{dom} g$ (see Proposition 6 below).

Proposition 3 *Let $f \in \Gamma(X)$ and $x^{**} \in X^{**}$. Then there exists a net $(x_i)_{i\in I} \subset X$ such that*

$$\widehat{x_i} \to^{w^*} x^{**}, \quad \|x_i\| \to \left\|x^{**}\right\|, \quad f(x_i) \to f^{**}(x^{**}).$$

An immediate consequence of this result is the relation

$$J(\mathrm{dom}\, f) \subset \mathrm{dom}\, f^{**} \subset \overline{J(\mathrm{dom}\, f)}^{w^*} \quad \forall f \in \Gamma(X).$$

Let us mention first which is the biconjugate of Af.

Proposition 4 (i) *If $f \in \Lambda(X)$ and $A \in L(X, Y)$ are such that* $\mathrm{dom}\, f^* \cap \mathrm{Im}\, A^* \neq \emptyset$ *then*

$$(Af)^{**} = \overline{A^{**} f^{**}}^{w^*}.$$

Moreover, if $0 \in {}^{ic}(\mathrm{dom}\, f^* - \mathrm{Im}\, A^*)$ then $(Af)^{**} = A^{**} f^{**}$ and the infimum in the definition of $A^{**} f^{**}$ is attained.

(ii) *If $f_1, \ldots, f_n \in \Lambda(X)$ are such that $\cap_{i \in \overline{1,n}}\, \mathrm{dom}\, f_i^* \neq \emptyset$ then*

$$(f_1 \square \cdots \square f_n)^{**} = \overline{f_1^{**} \square \cdots \square f_n^{**}}^{w^*}.$$

Moreover, if $n = 2$ and $0 \in {}^{ic}(\mathrm{dom}\, f_1^* - \mathrm{dom}\, f_2^*)$ then $(f_1 \square f_2)^{**} = f_1^{**} \square f_2^{**}$ and the second convolution is exact.

(iii) *If $f_1, \ldots, f_n \in \Gamma(X)$ are such that $\overline{f_1 \Diamond \cdots \Diamond f_n}$ is proper then*

$$(f_1 \Diamond \cdots \Diamond f_n)^{**} = \overline{f_1^{**} \Diamond \cdots \Diamond f_n^{**}}^{w^*}.$$

(When writing $\overline{f_1 \Diamond \cdots \Diamond f_n}$ we mean the closure is taken w.r.t. the norm topology.)

Proof (i) We have seen that $(Af)^* = f^* \circ A^*$ without any condition on f and A. Applying now Proposition 1 (i) for f^* and A^* for the dual system (X^*, X^{**}) we get the conclusion. For the second part apply [9, Th. 2.8.3 (vii)] for the Banach space X^* the operator $A^* \in L(Y^*, X^*)$, 0 and f^*. The assertion (ii) is obtained similarly.

(iii) We have seen that $f_1 \Diamond \cdots \Diamond f_n = Ak$, where k and A are defined in (3) and (4). By [9, Cor. 2.8.12] we have that

$$k^*\left(x_1^*, \ldots, x_n^*\right) = \min \left\{ \sum_{i=1}^{n} (\lambda_i f_i)^*(x_i^*) \mid (\lambda_1, \ldots, \lambda_n) \in \Delta_n \right\}$$

for every $\left(x_1^*, \ldots, x_n^*\right) \in (X^*)^n$, where

$$\Delta_n := \left\{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}_+)^n \mid \lambda_1 + \ldots + \lambda_n = 1\right\}.$$

It follows that $(f_1 \Diamond \cdots \Diamond f_n)^* = k^* \circ A^*$, and so

$$(f_1 \Diamond \cdots \Diamond f_n)^* = \min \left\{ \sum_{i=1}^{n} (\lambda_i f_i)^* \mid (\lambda_1, \ldots, \lambda_n) \in \Delta_n \right\}. \tag{8}$$

Applying this formula for $f_1^{**}, \ldots, f_n^{**}$ and the dual system (X^*, X^{**}) we get

$$(f_1^{**} \Diamond \cdots \Diamond f_n^{**})^{\star} = \min \left\{ \sum_{i=1}^{n} (\lambda_i f_i^{**})^{\star} \mid (\lambda_1, \ldots, \lambda_n) \in \Delta_n \right\}. \tag{9}$$

Using (5) and (6) for the dual systems (X, X^*) and (X^*, X^{**}), for all $\lambda \in \mathbb{R}_+ := [0, \infty)$ and $f \in \Gamma(X)$ we have $(\lambda f)^* = f^* \lambda = (f^{**})^{\star} \lambda = (\lambda f^{**})^{\star}$. From (8) and (9) we get $(f_1 \Diamond \cdots \Diamond f_n)^* = (f_1^{**} \Diamond \cdots \Diamond f_n^{**})^{\star}$. Taking the conjugate we get

$$(f_1 \Diamond \cdots \Diamond f_n)^{**} = (f_1^{**} \Diamond \cdots \Diamond f_n^{**})^{\star *} = \overline{f_1^{**} \Diamond \cdots \Diamond f_n^{**}}^{w^*}$$

because $\overline{f_1 \Diamond \cdots \Diamond f_n}$ is proper. □

Note that condition $\operatorname{dom} f^* \cap \operatorname{Im} A^* \neq \emptyset$ means that there exist $y^* \in Y^*$ and $\gamma \in \mathbb{R}$ such that $f(x) \geq \langle Ax, y^* \rangle + \gamma$ for every $x \in X$ (see [9, Cor. 2.6.4] for such a condition), while the condition $\cap_{i \in \overline{1,n}} \operatorname{dom} f_i^* \neq \emptyset$ means that there exist $x^* \in X^*$ and $\gamma \in \mathbb{R}$ such that $f_i(x) \geq \langle x, x^* \rangle + \gamma$ for all $i \in \overline{1,n}$ and $x \in X$ (see [9, Cor. 2.6.5] for such a condition). Moreover, in (i) and (ii) we can take arbitrary proper functions $f, f_1, \ldots, f_n$ satisfying these conditions.

We do not know if the formula below for $f_1 \Diamond \cdots \Diamond f_n$ is known.

Proposition 5 *Let* $f_1, \cdots, f_n \in \Lambda(X)$. *Then*

$$f_1 \Diamond \cdots \Diamond f_n = \max_{(\lambda_1,\ldots,\lambda_n) \in \Delta_n} [(\lambda_1 f_1) \square \ldots \square (\lambda_n f_n)]. \tag{10}$$

Proof It is clear that $\operatorname{dom} f_1 \Diamond \cdots \Diamond f_n = \operatorname{dom} f_1 + \ldots + \operatorname{dom} f_n$. Let $x \in \operatorname{dom} f_1 + \ldots + \operatorname{dom} f_n$ be fixed. Then $(f_1 \Diamond \cdots \Diamond f_n)(x) = v(P)$, the value of the problem (P), where (P) is the problem of minimizing t s.t. $f_i(u_i) - t \leq 0$ for $1 \leq i \leq n-1$, $f_n(x - u_1 - \ldots - u_n) - t \leq 0$. This is a convex problem verifying the Slater condition (just take $t > f_1(u_1) \vee \cdots \vee f_n(u_n)$, where $u_i \in \operatorname{dom} f_i$, for $1 \leq i \leq n$ with $u_1 + \ldots + u_n = x$). Applying [9, Th. 2.9.3] we obtain that $v(P) = v(D)$ and (D) has optimal solutions, where (D) is the dual problem of (P), that is, (D) is the problem of maximizing

$$\inf_{(u,t) \in X^{n-1} \times \mathbb{R}} \left[t + \sum_{i=1}^{n-1} \lambda_i (f_i(u_i) - t) + \lambda_n (f_n(x - u_1 - \cdots - u_{n-1}) - t) \right]$$

s.t. $\lambda_i, \ldots, \lambda_n \in \mathbb{R}_+$. Of course, here as everywhere throughout the paper, $0 \cdot \infty = \infty$. But

$$\inf_{(u,t) \in X^{n-1} \times \mathbb{R}} \left[t + \sum_{i=1}^{n-1} \lambda_i (f_i(u_i) - t) + \lambda_n (f_n(x - u_1 - \ldots - u_{n-1}) - t) \right] = -\infty$$

for $\lambda_i + \ldots + \lambda_n \neq 1$. Hence

$$(f_1 \Diamond \cdots \Diamond f_n)(x) = \max_{(\lambda_1,\ldots,\lambda_n) \in \Delta_n} \inf_{u \in X^{n-1}} \left[\sum_{i=1}^{n-1} \lambda_i f_i(u_i) + \lambda_n f_n(x - u_1 - \ldots - u_{n-1}) \right],$$

that is, (10) holds for $x \in \operatorname{dom} f_1 \Diamond \cdots \Diamond f_n$. Since (10) holds trivially for $x \notin \operatorname{dom} f_1 \Diamond \cdots \Diamond f_n$, the conclusion follows. □

4 The biconjugates of $h \circ A$ and $f + g$

Concerning the biconjugate of $h \circ A$ and $f + g$ we have the following result.

Proposition 6 (i) *Assume that* $h \in \Lambda(Y)$ *and* $A \in L(X, Y)$ *are such that* $\operatorname{dom} h \cap \operatorname{Im} A \neq \emptyset$. *Then*

$$(h \circ A)^{**} \geq h^{**} \circ A^{**}.$$

Moreover, if $h \in \Gamma(Y)$ *then*

$$(h \circ A)^{**} = h^{**} \circ A^{**} \iff \overline{A^* h^*}^{w^*} = \overline{A^* h^*} \iff (h \circ A)^* = \overline{A^* h^*}.$$

If either (a) *h is continuous at Ax for some* $x \in A^{-1}(\operatorname{dom} h)$, *or* (b) *$X$ and Y are complete,* $h \in \Gamma(Y)$ *and* $0 \in {}^{ic}(\operatorname{dom} h - \operatorname{Im} A)$, *then*

$$(h \circ A)^{**} = h^{**} \circ A^{**}. \tag{11}$$

(ii) *Assume that* $f_1, \ldots, f_n \in \Gamma(X)$ *are such that* $\cap_{i\in\overline{1,n}} \operatorname{dom} f_i \neq \emptyset$. *Then*

$$(f_1 + \cdots + f_n)^{**} \geq f_1^{**} + \cdots + f_n^{**},$$

with equality if and only if $\overline{f_1^* \square \cdots \square f_n^*}^{w^*} = \overline{f_1^* \square \cdots \square f_n^*}$. *Moreover, if* $n = 2$ *and either* f_1 *is continuous at some element in* $\operatorname{dom} f_1 \cap \operatorname{dom} f_2$*, or* X *is complete and* $0 \in {}^{ic}(\operatorname{dom} f_1 - \operatorname{dom} f_2)$ *then*

$$(f_1 + f_2)^{**} = f_1^{**} + f_2^{**}.$$

Proof (i) As seen in Proposition 1 (i) we have that

$$(h \circ A)^* = \overline{A^* h^*}^{w^*} \leq \overline{A^* h^*} \leq A^* h^*.$$

Hence

$$(h \circ A)^{**} = \left(\overline{A^* h^*}^{w^*}\right)^* \geq \left(\overline{A^* h^*}\right)^* = \left(A^* h^*\right)^* = h^{**} \circ A^{**}.$$

This relation also shows that (11) holds when $\overline{A^* h^*}^{w^*} = \overline{A^* h^*}$. Assume now that (11) holds, and so $\left(\overline{A^* h^*}^{w^*}\right)^* = \left(\overline{A^* h^*}\right)^*$. It follows that $\left(\overline{A^* h^*}^{w^*}\right)^{*\star} = \left(\overline{A^* h^*}\right)^{*\star}$. Because $\overline{A^* h^*}^{w^*}, \overline{A^* h^*} \in \Gamma_s(X^*)$, where s is the norm topology on X^*, using the biconjugate theorem we obtain that $\overline{A^* h^*}^{w^*} = \overline{A^* h^*}$. If h is continuous at Ax for some $x \in A^{-1}(\operatorname{dom} h)$, by a well-known result (see f.i. [9, Th. 2.8.3 (iii)]) we have that $(h \circ A)^* = A^* h^*$ (even with the infimum attained in the definition of $A^* h^*$). The same relation is true if X is complete and $0 \in {}^{ic}(\operatorname{dom} h - \operatorname{Im} A)$ (see [9, Th. 2.8.3 (vii)]). Hence $(h \circ A)^{**} = (A^* h^*)^* = h^{**} \circ A^{**}$.

(ii) The first part follows from (i) taking h and A as in (3) and (4), respectively. For the second part use, f.i., [9, Th. 2.8.7 (iii)] and [9, Th. 2.8.7 (vii)], respectively. □

Note that when $h \in \Lambda(Y)$ is continuous at some $Ax \in \operatorname{dom} h$ we have that $(h \circ A)^* = A^* h^*$, and so (11) holds without the semicontinuity of h. Similarly, no need of the semicontinuity of f_1 and f_2 when f_1 is continuous at some $x \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.

In [10, p. 212] it is given an example of functions $f_1, f_2 \in \Gamma(X)$ with $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \emptyset$ for X an arbitrary nonreflexive normed vector space with the property $\overline{f_1^* \square f_2^*}^{w^*} \neq \overline{f_1^* \square f_2^*}$; in fact f_1 and f_2 are indicator functions. Hence for these functions one has $(f_1 + f_2)^{**} \neq f_1^{**} + f_2^{**}$.

5 The biconjugate of $f \vee g$

Note first that from (5) one gets immediately

$$(\alpha f)^{**} = \alpha f^{**} \quad \forall f \in \Lambda(X),\ \forall \alpha \in \mathbb{P}, \tag{12}$$

but the relation is not true for $\alpha = 0$ even for $\dim X = 1$. Indeed, take $f(x) := (1 - x^2)^{-1}$ for $x \in (-1, 1)$, $f(x) := \infty$ otherwise. In fact, for $\alpha = 0$ we have

$$(0f)^{**} = (\iota_{\operatorname{dom} f})^{**} = \iota_{\overline{J(\operatorname{dom} f)}^{w^*}} = \iota_{\overline{\operatorname{dom} f^{**}}^{w^*}} \leq 0 f^{**} \quad \forall f \in \Gamma(X). \tag{13}$$

For this use (7) and Proposition 3.

In the next result we consider the case of the maximum $f \vee g$ of two convex functions.

Proposition 7 *Let* $f, g \in \Lambda(X)$ *with* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then*

$$(f \vee g)^{**} = \sup\left\{(\lambda f + \mu g)^{**} \mid (\lambda, \mu) \in \Delta_2\right\} \geq f^{**} \vee g^{**}. \tag{14}$$

Moreover, if either (a) f *is continuous at some point in* $\operatorname{dom} f \cap \operatorname{dom} g$ *and* $\overline{g}$ *is proper, or* (b) X *is complete,* $f, g \in \Gamma(X)$ *and* $0 \in {}^{ic}(\operatorname{dom} f - \operatorname{dom} g)$*, then*

$$(f \vee g)^{**} = f^{**} \vee g^{**}. \tag{15}$$

Proof By [9, Cor. 2.8.11] we have that

$$(f \vee g)^{*} = \min\left\{(\lambda f + \mu g)^{*} \mid (\lambda, \mu) \in \Delta_2\right\}.$$

The equality in (14) follows now using (1). Since $f \vee g \geq f$ we get $(f \vee g)^{**} \geq f^{**}$, and so the inequality in (14) holds true, too.

Assume now that one of the two mentioned conditions holds. Then, by [9, Cor. 2.8.13] one has

$$(f \vee g)^{*} = \min_{(\lambda,\mu)\in\Delta_2} \left[(\lambda f)^{*} \Box (\mu g)^{*}\right],$$

and so, using again (1) as well as relations (12), (2) and (13), we get

$$\begin{aligned}
(f \vee g)^{**} &= \sup_{(\lambda,\mu)\in\Delta_2} \left[(\lambda f)^{*} \Box (\mu g)^{*}\right]^{*} = \sup_{(\lambda,\mu)\in\Delta_2} \left[(\lambda f)^{**} + (\mu g)^{**}\right] \\
&= \max\left\{ \sup_{\lambda\in(0,1)} \left[\lambda f^{**} + (1-\lambda) g^{**}\right], \iota_{\operatorname{dom} f}^{**} + g^{**}, f^{**} + \iota_{\operatorname{dom} g}^{**} \right\} \\
&= \max\left\{ f^{**} \vee g^{**}, \iota_{\overline{\operatorname{dom} f^{**}}^{w^*}} + g^{**}, f^{**} + \iota_{\overline{\operatorname{dom} g^{**}}^{w^*}} \right\};
\end{aligned}$$

it was possible to apply (2) because f^{**} and g^{**} are proper functions under the present conditions. If $(f^{**} \vee g^{**})(x^{**}) = \infty$ then clearly $(f \vee g)^{**}(x^{**}) = (f^{**} \vee g^{**})(x^{**})$. Let $x^{**} \in X^{**}$ be such that $f^{**}(x^{**}) \leq g^{**}(x^{**}) < \infty$. Then

$$\left(f^{**} + \iota_{\overline{\operatorname{dom} g^{**}}^{w^*}}\right)(x^{**}) \leq \left(\iota_{\overline{\operatorname{dom} f^{**}}^{w^*}} + g^{**}\right)(x^{**}) = \left(f^{**} \vee g^{**}\right)(x^{**}),$$

and so the conclusion follows. □

The condition $\overline{g}$ is proper in (a) is essential even in Hilbert spaces. To see this consider X infinite dimensional, $f := 0$ and $g(x) := \varphi(x)$ for $x \in H := \{u \in X \mid \langle u, x^* \rangle \geq 0\}$ and $g(x) := \infty$ for $x \notin H$, where $x^* \in X^* \setminus \{0\}$ and $\varphi \in X' \setminus X^*$. Then $f^{**} = 0$ and $g^{**} = -\infty$, and so $f^{**} \vee g^{**} = f^{**} = 0$. On the other hand $(f \vee g)(x) = 0 \vee \varphi(x) \geq \iota_H(x)$ for $x \in X$. Hence $\overline{f \vee g} \geq \iota_H$. Since φ is not continuous, φ is not bounded below on any nonempty open set. Therefore, for $x \in X$ with $\langle x, x^* \rangle < 0$ there exists a sequence (x_n) converging to x such that $\varphi(x_n) \to -\infty$. It follows that $\overline{f \vee g}(x) \leq 0$, and so $\overline{f \vee g} = \iota_H$. Hence $(f \vee g)^{**} = \iota_H^{**} = \iota_{\overline{H}^{w^*}} \neq 0$ because $\overline{H}^{w^*} = \{x^{**} \in X^{**} \mid \langle x^*, x^{**} \rangle \geq 0\}$.

Note that the main aim of [3] was to deduce the formula (15); it was obtained under condition (b) of Proposition 7 (see [3, Th. 6]) with a quite involved proof. For another approach see [1].

In [3] the authors gave also a formula for the preconjugate of $f^* \vee g^*$ with $f, g \in \Gamma(X)$ and $\operatorname{dom} f^* \cap \operatorname{dom} g^* \neq \emptyset$, that is, for $(f^* \vee g^*)^{\star}$. This is the function defined in [3, p. 3554] and denoted by $f \underset{0}{\wedge} g$, the expression of which is quite complicated. Below we give another expression of the preconjugate of $f^* \vee g^*$.

Proposition 8 *Let X be a locally convex space.*

(i) *Let $f \in \Lambda(X)$ and $\varphi_f : \mathbb{R} \times X \to \overline{\mathbb{R}}$, $\varphi_f(\alpha, \cdot) := f\alpha$ for $\alpha \in \mathbb{P}$ and $\varphi_f(\alpha, \cdot) := \infty$ otherwise. Then $\varphi_f \in \Lambda(\mathbb{R} \times X)$ and φ_f is positively homogeneous.*

(ii) *Let* $f \in \Gamma(X)$ *and* $\overline{\varphi}_f : \mathbb{R} \times X \to \overline{\mathbb{R}}$, $\overline{\varphi}_f(\alpha, \cdot) := f\alpha$ *for* $\alpha \in \mathbb{R}_+$ *and* $\overline{\varphi}_f(\alpha, \cdot) := \infty$ *otherwise. Then* $\overline{\varphi}_f \in \Gamma(\mathbb{R} \times X)$ *and* $\overline{\varphi}_f$ *is positively homogeneous. Moreover,* $\overline{\varphi}_f = \overline{\varphi_f}$.

(iii) *Let* $f \in \Gamma(X)$ *and* $x \in X$. *If* $f(0) \le 0$ *then* $\overline{\varphi}_f(\cdot, x)$ *is nonincreasing and right-continuous; if* $f \ge f_\infty$ *then* $\overline{\varphi}_f(\cdot, x)$ *is nondecreasing on* $\mathbb{R}_+$.

(iv) *If* $f, g \in \Lambda(X)$, *then the function*

$$f \diamond g := \inf\{(f\lambda)\Box(g\mu) \mid \lambda, \mu > 0,\ \lambda + \mu = 1\}$$

is convex. Moreover, if $f, g \in \Gamma(X)$ *and* $\operatorname{dom} f^* \cap \operatorname{dom} g^* \ne \emptyset$ *then*

$$\overline{f \diamond g} \le f \,\bar{\diamond}\, g := \inf\{(f\lambda)\Box(g\mu) \mid (\lambda, \mu) \in \Delta_2\} \le f \diamond g,$$

$f \,\bar{\diamond}\, g \in \Lambda(X)$, $\overline{f \diamond g} \in \Gamma(X)$ *and* $\left(\overline{f \diamond g}\right)^* = (f \,\bar{\diamond}\, g)^* = (f \diamond g)^* = f^* \vee g^*$.

Proof (i) It is easy to see that

$$\operatorname{epi} \varphi_f = \left\{(\alpha, u, t) \mid \alpha > 0,\ (\alpha^{-1}u, \alpha^{-1}t) \in \operatorname{epi} f\right\} = \mathbb{P} \cdot (\{1\} \times \operatorname{epi} f).$$

Since $\operatorname{epi} f$ is convex, $\operatorname{epi} \varphi_f$ is also convex. Hence φ_f is convex. The positive homogeneity of φ_f is immediate from its definition.

(ii) Having a nonempty closed convex set $A \subset X$ we have that

$$\operatorname{cl}(\mathbb{P}(\{1\} \times A)) = \operatorname{cl}(\mathbb{R}_+(\{1\} \times A)) = (\{0\} \times A_\infty) \cup (\mathbb{P}(\{1\} \times A)).$$

Hence, for $A := \operatorname{epi} f$ with $f \in \Gamma(X)$, we have

$$\operatorname{epi} \overline{\varphi}_f = (\{0\} \times A_\infty) \cup (\mathbb{P}(\{1\} \times A)) = \operatorname{cl}(\operatorname{epi} \varphi_f)$$

(iii) Let $f \in \Gamma(X)$. Assume first that $f(0) \le 0$ and fix $x \in X$. Since the mapping $\mathbb{P} \ni s \longmapsto s^{-1}[f(0 + sx) - f(0)]$ is nondecreasing and $f(0) \le 0$ we obtain that $\mathbb{P} \ni s \longmapsto s^{-1} f(sx)$ is nondecreasing as the sum of two nondecreasing functions. It follows that $\overline{\varphi}_f(\cdot, x)$ is nonincreasing. This implies that $I := \operatorname{dom} \overline{\varphi}_f(\cdot, x)$ is an interval included in $\mathbb{R}_+$ with $\sup I = \infty$ when $I \ne \emptyset$. Because $\overline{\varphi}_f$ is lsc we obtain that $\overline{\varphi}_f(\cdot, x)$ is right-continuous (and even continuous on $\mathbb{R} \setminus \{\inf I\}$.

Assume now that $f \ge f_\infty$. Then $\operatorname{epi} f \subset \operatorname{epi} f_\infty = (\operatorname{epi} f)_\infty$. Let $x \in \operatorname{dom} f$ and $1 < t$. Then $t(x, f(x)) = (x, f(x)) + (t-1)(x, f(x)) \in \operatorname{epi} f + (\operatorname{epi} f)_\infty \subset \operatorname{epi} f$, and so $f(tx) \le tf(x)$. Hence $f(tx) \le tf(x)$ for all $x \in X$ and $t \in [1, \infty)$. Taking now $0 < s < t$ and $x \in X$ we get

$$f_\infty(x) = sf_\infty(s^{-1}x) \le sf(s^{-1}x) = sf\left(ts^{-1}t^{-1}x\right) \le sts^{-1} f(t^{-1}x) = tf(t^{-1}x),$$

which means that $(f0)(x) \le (fs)(x) \le (ft)(x)$. Hence $\overline{\varphi}_f(x, \cdot)$ is nondecreasing on $\mathbb{R}_+$.

(iv) Let $f, g \in \Lambda(X)$. From (i) we obtain that the mapping $F : X \times X \times \mathbb{R} \to \overline{\mathbb{R}}$ defined by $F(x, u, t) := \varphi_f(u, t) + \varphi_g(x - u, 1 - t)$ is convex. Since $(f \diamond g)(x) = \inf_{(u,t) \in X \times \mathbb{R}} F(x, u, t)$, $f \diamond g$ is convex, too (as a marginal function associated to a convex function). Moreover, using (1), (5) and (2) we get

$$\begin{aligned}(f \diamond g)^* &= \sup\left\{[(f\lambda)\Box(g\mu)]^* \mid \lambda, \mu > 0,\ \lambda + \mu = 1\right\}\\ &= \sup\left\{(f\lambda)^* + (g\mu)^* \mid \lambda, \mu > 0,\ \lambda + \mu = 1\right\}\\ &= \sup\left\{\lambda f^* + \mu g^* \mid \lambda, \mu > 0,\ \lambda + \mu = 1\right\} = f^* \vee g^*.\end{aligned}$$

Assuming that $f, g \in \Gamma(X)$ and taking $\widetilde{F}(x, u, t) := \overline{\varphi}_f(u, t) + \overline{\varphi}_g(x - u, 1 - t)$, we obtain, as for $f \diamond g$, that $f \,\bar{\diamond}\, g$ is convex, and, of course, $f \,\bar{\diamond}\, g \le f \diamond g$. A similar computation as for $(f \diamond g)^*$ yields

$$(f \bar{\diamond} g)^* = \max\left\{f^* \vee g^*, \iota_{\overline{\mathrm{dom}\, f^*}^{w^*}} + g^*, f^* + \iota_{\overline{\mathrm{dom}\, g^*}^{w^*}}\right\},$$

whence, as in the proof of the preceding proposition, we get $(f \bar{\diamond} g)^* = f^* \vee g^*$. Assuming, moreover, that $\mathrm{dom}\, f^* \cap \mathrm{dom}\, g^* \neq \emptyset$, $f^* \vee g^* \in \Gamma_{w^*}(X^*)$, and so $\overline{f \diamond g} = (f \diamond g)^{**} = (f^* \vee g^*)^\star \in \Gamma(X)$. The conclusion follows. □

The arguments above show that for $f, g \in \Lambda(X)$ with $\mathrm{dom}\, f^* \cap \mathrm{dom}\, g^* \neq \emptyset$ we have $\overline{f \diamond g} \in \Gamma(X)$ and $(\overline{f \diamond g})^* = f^* \vee g^*$. Note that the definition of $f \underset{0}{\wedge} g$ in [3] is specific to normed vector spaces; more precisely, $f \underset{0}{\wedge} g = \lim_{\delta \searrow 0} f \underset{\delta}{\wedge} g$, and the expression of $f \underset{\delta}{\wedge} g$ involves the norm. Assuming that X is a nvs, we obtain that $f \underset{0}{\wedge} g = \overline{f \diamond g}$ whenever $f, g \in \Gamma(X)$ with $\mathrm{dom}\, f^* \cap \mathrm{dom}\, g^* \neq \emptyset$. Moreover, as in [3, Th. 12], one obtains that for $f, g \in \Gamma(X)$ with $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$ one has

$$(f \vee g)^{**} = f^{**} \vee g^{**} \iff (f \vee g)^* = \overline{f^* \diamond g^*}.$$

Note that the function $\overline{\varphi}_f$ was introduced in [7, Cor. 13.5.1]. Moreover, the first part of assertion (iii) is obtained in [6, Prop. 2.1] in finite dimensional spaces.

6 The biconjugate of $f \vartriangle g$

Let X, Y be locally convex spaces and $f \in \Gamma(X)$, $g \in \Gamma(Y)$. Following [6, Def. 2.1], we define a function $f \vartriangle g : X \times Y \to \overline{\mathbb{R}}$ for pairs (f, g) of types I and II. More precisely we say that

(i) the pair (f, g) is of type I if $f(0) \leq 0$ and $g(y) \leq 0$ for every $y \in \mathrm{dom}\, g$; in this case

$$(f \vartriangle g)(x, y) := \begin{cases} \overline{\varphi}_f(-g(y), x) & \text{if } x \in X,\ y \in \mathrm{dom}\, g, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\overline{\varphi}_f$ is defined in Proposition 8 (ii);

(ii) the pair (f, g) is of type II if $f \geq f_\infty$ and $g \geq 0$; in this case, when $f \neq f_\infty$,

$$(f \vartriangle g)(x, y) := \begin{cases} \overline{\varphi}_f(g(y), x) & \text{if } x \in X,\ y \in \mathrm{dom}\, g, \\ +\infty & \text{otherwise,} \end{cases}$$

while if $f = f_\infty$,

$$(f \vartriangle g)(x, y) := f(x) + \iota_{\overline{\mathrm{dom}\, g}}(y).$$

First observe that for $f \in \Gamma(X)$ we have the following (almost) obvious equivalences (see also [6, Lem. 2.1]):

$$f(0) \leq 0 \iff f^* \geq 0, \tag{16}$$

$$f \geq f_\infty \iff f^* \leq (f_\infty)^* = \iota_{\overline{\mathrm{dom}\, f^*}^{w^*}} \iff f^*(x^*) \leq 0\ \forall x^* \in \mathrm{dom}\, f^*, \tag{17}$$

$$f = f_\infty \iff [f \geq f_\infty \text{ and } f(0) = 0] \iff f^* = \iota_{\mathrm{dom}\, f^*}, \tag{18}$$

as well as the following relation

$$\sup_{x \in [f < \gamma]} \left[\langle x, x^* \rangle - \eta f(x)\right] = \sup_{x \in [f \leq \gamma]} \left[\langle x, x^* \rangle - \eta f(x)\right] \quad \forall \eta \in \mathbb{R}_+,\ \forall \gamma > \inf f, \tag{19}$$

where $[f \leq \gamma] := \{x \in X \mid f(x) \leq \gamma\}$ and $[f < \gamma]$, $[f = \gamma]$ being defined similarly; for (19) fix some $\overline{x} \in [f < \gamma]$ and observe that for $x \in [f \leq \gamma]$ and $\lambda \in (0, 1)$ one has

$(1-\lambda)x+\lambda\overline{x} \in [f < \gamma]$ and take into account that the restriction of f to the segment $[\overline{x}, x]$ is continuous.

From (16)–(18) we obtain immediately that, (f, g) is of type I (resp. of type II) if and only if (g^*, f^*) is of type II (resp. of type I).

In the next result we prove that $f \vartriangle g \in \Gamma(X \times Y)$ whenever (f, g) is of type I. In a similar way one can obtain that $f \vartriangle g \in \Gamma(X \times Y)$ when (f, g) is of type II, but this will follow also from Proposition 9 below.

Lemma 1 *Let (f, g) be of type I. Then $f \vartriangle g \in \Gamma(X \times Y)$.*

Proof Using the fact $\overline{\varphi}_f$ is convex and $\overline{\varphi}_f(\cdot, x)$ is nonincreasing [see Proposition 8 (ii) and (iii)] one obtains easily that $f \vartriangle g \in \Lambda(X \times Y)$. Consider a net $((x_i, y_i, t_i))_{i\in I} \subset \operatorname{epi} f \vartriangle g$ converging to $(x, y, t) \in X \times Y \times \mathbb{R}$. It follows that $0 \geq \gamma_i := g(y_i)$ for every i. We may (and we do) assume that $\gamma_i \to \gamma \in [g(y), 0]$. If $\gamma_i = 0$ for $i \in J$ with $J \subset I$ cofinal, then $t_i \geq (f \vartriangle g)(x_i, y_i) = f_\infty(x_i)$, whence $t \geq f_\infty(x)$. Else we may assume that $\gamma_i > 0$ for every $i \in I$. Then $(-\gamma_i^{-1}x_i, -\gamma_i^{-1}t_i) \in \operatorname{epi} f$ for $i \in I$. If $\gamma = 0$, using the convexity of f and the fact that $(0, 0) \in \operatorname{epi} f$ we obtain rapidly that $(x, t) \in (\operatorname{epi} f)_\infty = \operatorname{epi} f_\infty$, and so $t \geq f_\infty(x)$. Since $\overline{\varphi}_f(\cdot, x)$ is nonincreasing we get $(f \vartriangle g)(x, y) = \overline{\varphi}_f(-g(y), x) \leq \overline{\varphi}_f(0, x) = f_\infty(x) \leq t$. If $\gamma \neq 0$, since $(-\gamma_i^{-1}x_i, -\gamma_i^{-1}t_i) \in \operatorname{epi} f$, we obtain that $(-\gamma^{-1}x, -\gamma^{-1}t) \in \operatorname{epi} f$. Using again the monotonicity of $\overline{\varphi}_f(\cdot, x)$, we get $(f \vartriangle g)(x, y) = \overline{\varphi}_f(-g(y), x) \leq \overline{\varphi}_f(-\gamma, x) \leq t$. Hence $(x, y, t) \in \operatorname{epi} f \vartriangle g$. □

The next result is stated in [6, Th. 2.1] for X and Y finite dimensional spaces. We provide its proof for reader's convenience.

Proposition 9 *Let $f \in \Gamma(X)$ and $g \in \Gamma(Y)$ be such that (f, g) is of type I or II. Then (g^*, f^*) is of type II or I, respectively, and $(f \vartriangle g)^*(x^*, y^*) = (g^* \vartriangle f^*)(y^*, x^*)$. Consequently, $f \vartriangle g \in \Gamma(X \times Y)$.*

Proof Fix $(x^*, y^*) \in X^* \times Y^*$ and set $\mu := (f \vartriangle g)^*(x^*, y^*)$, $\eta := (g^* \vartriangle f^*)(y^*, x^*)$.

(i) Assume that (f, g) is of type I. Using Lemma 1 we have that $f \vartriangle g \in \Gamma(X \times Y)$. Moreover, as observed above, (g^*, f^*) is of type II, that is, $g^* \geq (g^*)_\infty$ and $f^* \geq 0$. Assume first that $[g < 0] = \emptyset$. Then, by (17) and (18) applied for f replaced by g^*, we get $g^* = (g^*)_\infty$ and $\mu = \alpha$, where

$$\begin{aligned}\alpha &:= \sup\left\{\langle x, x^*\rangle + \langle y, y^*\rangle - f_\infty(x) \mid x \in X,\ y \in [g = 0]\right\} \\ &= \iota_{\overline{\operatorname{dom} f^*}^{w^*}}(x^*) + \iota^*_{\operatorname{dom} g}(y^*) = \iota_{\overline{\operatorname{dom} f^*}^{w^*}}(x^*) + (g^*)_\infty(y^*) = \eta.\end{aligned} \tag{20}$$

Assume now that $[g < 0] \neq \emptyset$. Then $\mu = \alpha \vee \beta$ with α defined by (20) and

$$\begin{aligned}\beta &:= \sup\left\{\langle x, x^*\rangle + \langle y, y^*\rangle + g(y)f\left((-g(y))^{-1}x\right) \mid x \in X,\ y \in [g < 0]\right\} \\ &= \sup\left\{-g(y)\langle x, x^*\rangle + \langle y, y^*\rangle + g(y)f(x) \mid x \in X,\ y \in [g < 0]\right\} \\ &= \sup\left\{\langle y, y^*\rangle - g(y)f^*(x^*) \mid y \in [g < 0]\right\}.\end{aligned}$$

Hence $\beta = \infty$ if $f^*(x^*) = \infty$. Since $\operatorname{dom} g = [g \leq 0]$, by (19), $\beta = \iota^*_{\operatorname{dom} g}(y^*) = (g^*)_\infty(y^*)$ when $f^*(x^*) = 0$ and $\beta = f^*(x^*)g^*\big((f^*(x^*))^{-1}\, y^*\big)$ when $f^*(x^*) \in \mathbb{P}$. Hence $\beta = \eta$. On the other hand, from (20) we have that

$$\alpha = \iota_{\overline{\operatorname{dom} f^*}^{w^*}}(x^*) + \iota^*_{[g=0]}(y^*) \leq \iota_{\overline{\operatorname{dom} f^*}^{w^*}}(x^*) + \iota^*_{\operatorname{dom} g}(y^*) = \iota_{\overline{\operatorname{dom} f^*}}(x^*) + (g^*)_\infty(y^*).$$

Hence $\alpha \le \beta$ if $f^*(x^*) = \infty$, $\alpha = \beta$ if $f^*(x^*) = 0$ and, because $(g^*)_\infty \le g^*$, $\alpha = (g^*)_\infty(y^*) = f^*(x^*)(g^*)_\infty\big((f^*(x^*))^{-1} y^*\big) \le \beta$ if $f^*(x^*) \in \mathbb{P}$. Therefore, $\mu = \eta$.

(ii) Assume now that (f, g) is of type II. Then, as seen above, (g^*, f^*) is of type I. By (i) we obtain that $g^* \vartriangle f^* \in \Gamma_{w^*}(Y^* \times X^*)$ and $(g^* \vartriangle f^*)^\star(y, x) = (f^{*\star} \vartriangle g^{*\star})(x, y) = (f \vartriangle g)(x, y)$ for all $(x, y) \in X \times Y$. It follows that $f \vartriangle g \in \Gamma(X \times Y)$ and

$$(f \vartriangle g)^*(x^*, y^*) = (g^* \vartriangle f^*)^{\star *}(y^*, x^*) = (g^* \vartriangle f^*)(y^*, x^*).$$

The proof is complete. □

Proposition 10 *Let X and Y be normed vector spaces and let $f \in \Gamma(X)$, $g \in \Gamma(Y)$ with (f, g) of type I or II. Then $(f \vartriangle g)^{**} = f^{**} \vartriangle g^{**}$.*

Proof Applying twice Proposition 9 we obtain the conclusion. □

References

1. Boţ, R.I., Wanka, G.: The conjugate of the pointwise maximum of two convex functions revisited, Preprint, Chemnitz University of Technology (2006)
2. Chakrabarty, A.K., Shunmugaraj, P., Zălinescu, C.: Continuity properties for the subdifferential and ε. J. Convex Anal. **14**, 479–514 (2007)
3. Fitzpatrick, S.P., Simons, S.: On the pointwise maximum of convex functions. Proc. Am. Math. Soc. **128**, 3553–3561 (2000)
4. Fitzpatrick, S.P., Simons, S.: The conjugates, compositions and marginals of convex functions. J. Convex Anal. **8**, 423–446 (2001)
5. Gossez, J.-P.: Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs. J. Math. Anal. Appl. **34**, 371–395 (1971)
6. Maréchal, P.: On a functional operation generating convex functions. I: Duality. J. Optim. Theory Appl. **126**, 175–189 (2005)
7. Rockafellar, R.T.: On the maximal monotonicity of subdifferential mappings. Pac. J. Math. **33**, 209–216 (1970)
8. Simons, S.: Minimax and monotonicity, Springer-Verlag, Berlin (1998).
9. Zălinescu, C.: Convex analysis in general vector spaces, World Scientific, Singapore (2002).
10. Zălinescu, C.: Slice convergence for some classes of convex functions. J. Nonlinear Convex Anal. **4**, 185–214 (2003)